

Tutte polynomial general factorization for graphs on general specializations

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Abstract

The general unique factorization formula for the Tutte polynomial, Negami's formula, is not valid on certain specializations of the plane. The easiest example of this fact is the spanning tree number; the specialization $x = y = 1$. We prove that the only specializations where Negami's formula doesn't hold are those corresponding to the spanning forest number generator, the q-state Potts model and particularizations of these. The Ising model is one of these specializations. We show the non unique factorization formulas for these specializations.

Keywords: Tutte polynomial, Graph Theory, Factorization

1. Introduction

The Tutte polynomial of a graph G , also known as dichromate or Tutte-Whitney polynomial, is defined as the following subgraph generating function [Tu]:

$$T_G(x, y) = \sum_{\substack{A \subseteq G \\ V(A)=V(G)}} (x-1)^{k(A)-k(G)} (y-1)^{k(A)+|E(A)|-|V(G)|}$$

It is the most general graph invariant that can be defined by the *deletion-contraction algorithm*:

$$T_G(x, y) = T_{G \cdot e}(x, y) + T_{G - e}(x, y)$$

$$\begin{aligned}
\text{Graph } G &= \frac{1}{(x-1)(y-1)-1} \left((y-1) \left(\text{Graph } G_1 \text{ with loop} - \text{Graph } G_1 \text{ with bridge} \right) \right. \\
&\quad \left. - \text{Graph } G_2 \text{ with loop} + (x-1) \left(\text{Graph } G_2 \text{ with bridge} \right) \right)
\end{aligned}$$

Figure 1: Tutte polynomial factorization for two sharing nodes

where e is neither a loop (and edge with coincident endpoints) nor a bridge (an edge whose deletion increases the number of connected components) such that $T_G(x, y) = x^i y^j$ if the edge set of G only has i bridges and j loops. Here $G \cdot e$ and $G - e$ denote the contraction and deletion of the edge e respectively (See Figures 6 and 7). Computing the Tutte polynomial is in general a NP-hard problem [Ja].

As far as the author knows, the first splitting formula for the Tutte polynomial was given in [Br] for a 2-sum graph: Consider a connected graph G with the property that it can be decomposed by connected subgraphs G_1 and G_2 only sharing a pair nodes a and b . Denoting by \hat{G}_i the graph resulting from the identification of the nodes a and b in the graph G_i , we have the following factorization formula for the Tutte polynomial:

$$\begin{aligned}
T_G(x, y) &= \frac{1}{(x-1)(y-1)-1} \left((y-1) T_{G_1}(x, y) T_{G_2}(x, y) - T_{G_1}(x, y) T_{\hat{G}_2}(x, y) \right. \\
&\quad \left. - T_{\hat{G}_1}(x, y) T_{G_2}(x, y) + (x-1) T_{\hat{G}_1}(x, y) T_{\hat{G}_2}(x, y) \right)
\end{aligned}$$

Figure 1 shows this factorization.

The splitting formula for the general case of a k -sum of graphs is Negami's formula [Ne]. A colored version of this formula was developed in [Tr]. As it was written at the end of [Tr], there are results regarding Tutte polynomials of generalized parallel connections of general matroids ([An] for the 3-sum and [BM] for the general case), but those generalized parallel connections are not general enough to include all k -sums of graphs ¹.

¹The author is grateful to Prof. Lorenzo Traldi and Prof. Anna de Mier for these

Specialization	Invariant
$xy = 1$	Jones polynomial
$y = 0$	Chromatic polynomial
$x = 1, y \neq 1$	Reliability polynomial
$x = 0$	Flow polynomial
$(x - 1)(y - 1) = 2$	Ising model
$(x - 1)(y - 1) = q$	q-state Potts model
$y \neq 1$	Random cluster model
$y = 1$	Spanning forest number generator
$(1, 1)$	Spanning tree number
$(2, 1)$	Spanning forest number
$(1, 2)$	Spanning subgraph number

Table 1: Specializations of the Tutte polynomial

Different specializations of the Tutte polynomial naturally appear as classical invariants in several branches of mathematics, physics and engineering [Bo], [Bi],[BO]. For example, the Jones polynomial in knot theory [Jo], the chromatic polynomial in topology, the reliability polynomial in network engineering, the Ising and Potts model in statistical mechanics [Is], [On], [Po], the random cluster model [FK], etc. (See Table 1). This fact make us wonder about a splitting formula for the Tutte polynomial in different specializations. This is not a trivial question. For example, none of the splitting formulas mentioned before apply to the spanning tree number, the specialization $x = y = 1$.

The factorization coefficients of Negami's splitting formula are rational functions in the field of fractions $\mathbb{Q}(x, y)$. A general element of this field could have problems under certain specializations. We show this problem with an easy example: Consider the element $a(x, y) = 1/(y - x^2)$. The specializations along vertical and horizontal lines are perfectly defined in the respective field of fractions:

$$\frac{1}{y-x^2} \in \mathbb{Q}(x, y) \xrightarrow{x=x_0} \frac{1}{y-x_0^2} \in \mathbb{Q}(y)$$

references and valuable comments.

$$\frac{1}{y-x^2} \in \mathbb{Q}(x, y) \xrightarrow{y=y_0} \frac{1}{y_0-x^2} \in \mathbb{Q}(x)$$

We can also specialize $a(x, y)$ along a parameterized curve giving an element in the field of fractions on the parameter:

$$\frac{1}{y-x^2} \in \mathbb{Q}(x, y) \xrightarrow{x=t^2, y=t^3} \frac{1}{t^3-t^4} \in \mathbb{Q}(t)$$

However, there is **no** specialization along the parabola $y = x^2$:

$$\frac{1}{y-x^2} \in \mathbb{Q}(x, y) \xrightarrow{y=x^2} \nexists$$

The same problem occurs with the Tutte polynomial factorization coefficients and it is responsible for the **non** uniqueness of the factorization for certain specializations and number of sharing nodes.

Recently, we proved a general factorization formula for the all terminal network reliability of a stochastic graph G [BR]. In particular we have a general factorization formula for the Tutte polynomial along the line $x = 1$, $y \neq 1$. Following the same line of ideas, in section 4 we rederive Negami's formula; i.e. the factorization theorem treating the Tutte polynomial $T_G(x, y)$ as an element of the polynomial ring $\mathbb{Z}[x, y]$ and the factorization coefficients as elements of the field of fractions $\mathbb{Q}(x, y)$ (there is no specialization):

Theorem 1.1. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_G(x, y) = \sum_{A, B \in \text{Part}_n} c_{AB}(x, y) T_{G_1^A}(x, y) T_{G_2^B}(x, y)$$

such that:

$$c_{AB}(x, y) = b_{AB}((x-1)(y-1)) (y-1)^{|A|+|B|-n-1}$$

where $(b_{AB}(t))_{A, B \in \text{Part}_n}$ is the inverse of the polynomial connectivity matrix:

$$(t^{|A \cdot B|-1})_{A, B \in \text{Part}_n}$$

Here, each graph G_i has n distinguished vertices, the sharing vertices, and $G_i^{\mathcal{A}}$ denote the identification of the distinguished vertices that belong to the same class in the partition $\mathcal{A} \in \text{Part}_n$ of the distinguished vertices set (See Figure 4). We denote by $|\mathcal{A}|$ the number of classes in the partition \mathcal{A} and define the product $\mathcal{A} \cdot \mathcal{B}$ as the finer partition coarser than \mathcal{A} and \mathcal{B} .

The rederivation of Negami's formula via an extended notion of network reliability is a novel feature of the paper. Section 3 defines the *extended reliability* and gives a formula for its k -sum factorization. As a Corollary of this reliability formula, in section 4 we get Negami's formula.

In section 7 we prove the following new determinant formula:

$$\det \left((t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \right) = \prod_{\mathcal{A} \in \text{Part}_n} \prod_{\lambda=1}^{|\mathcal{A}|-1} (t - \lambda) \quad (1)$$

This is a major improvement on the determinant formula proved in [Bu]. The proof is technical and similar to the one in [Bu]. In particular, it shows that there is **no** specialization of the factorization matrix $(c_{\mathcal{A}\mathcal{B}}(x, y))$ on the curves:

$$(x - 1)(y - 1) - \lambda = 0, \quad \lambda = 1, 2, \dots, n - 1$$

and $y = 1$ if $n \geq 4$ (this is not obvious) for n sharing nodes. Moreover, these are the only regions where Negami's formula doesn't apply.

In section 6 we prove that there is a **non** trivial affine space of factorizations on these particular specializations. For the curve $(x - 1)(y - 1) - \lambda = 0$ and their particular points such that $\lambda = 1, 2, \dots, n - 1$ we have:

Theorem 1.2. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_{G, \lambda}(x, y) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} c_{\mathcal{A}\mathcal{B}}(x, y) T_{G_1^{\mathcal{A}}, \lambda}(x, y) T_{G_2^{\mathcal{B}}, \lambda}(x, y)$$

where the subscript λ denotes the restriction of the Tutte polynomial in the region $(x - 1)(y - 1) - \lambda = 0$, such that:

$$c_{\mathcal{A}\mathcal{B}}(x, y) = b_{\mathcal{A}\mathcal{B}}(\lambda) (y - 1)^{|\mathcal{A}| + |\mathcal{B}| - n - 1} \quad (2)$$

where $(b_{\mathcal{A}\mathcal{B}}(t))_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \in \mathcal{M}_{n, t}$ is a solution of the equation $A(t)B(t)A(t) = A(t)$ where $A(t) = (t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n}$.

n\t	1	2	3	4	5	6
1						
2	3					
3	24	9				
4	224	161	29			
5	2703	2448	1023	103		
6	41208	40185	26325	6240	405	
7	769128	765033	635904	257904	38104	1753

Table 2: Dimension of the affine space $\mathcal{M}_{n,t}$

Table 2 shows the dimension of the respective affine spaces $\mathcal{M}_{n,t}$. For the curve $y = 1$ and their particular points we have:

Theorem 1.3. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_G(x, 1) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} c_{\mathcal{A}\mathcal{B}}(x, 1) T_{G_1^{\mathcal{A}}}(x, 1) T_{G_2^{\mathcal{B}}}(x, 1)$$

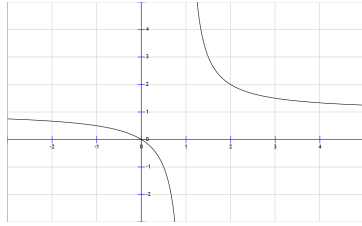
such that $C(x) = (c_{\mathcal{A}\mathcal{B}}(x, 1))$ is a solution of the equation $A(x)C(x)A(x) = A(x)$ where

$$A(x) = (a_{\mathcal{A}\mathcal{B}}(x, 1)) = ((x-1)^{|\mathcal{A}\mathcal{B}|-1} \delta_{n+|\mathcal{A}\mathcal{B}|-|\mathcal{A}|-|\mathcal{B}|})$$

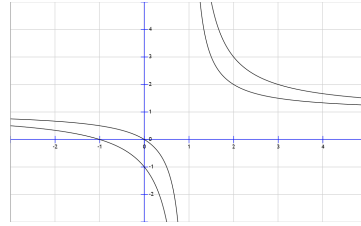
Moreover, the equation $A(x)C(x)A(x) = A(x)$ has a unique solution $C(x)$ if and only if $n \leq 3$.

Formula 1 and Theorems 1.2 and 1.3 constitute the main result of the paper. The following Corollary is illustrated in Figure 2.

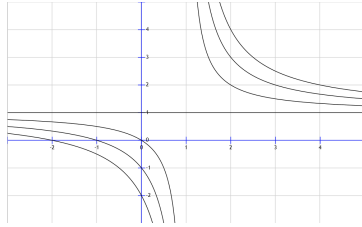
Corollary 1.4. *There is no Tutte polynomial unique general factorization for n sharing nodes only in the specialization to the curve $(x-1)(y-1)-\lambda = 0$ and their particular points such that $\lambda = 1, 2, \dots, n-1$ and the curve $y = 1$ and their particular points if $n \geq 4$; i.e. The spanning forest number generator, the q -state Potts model and their particularizations are the only models with no unique general factorization for $n \geq q + 1$ and $n \geq 4$ respectively.*



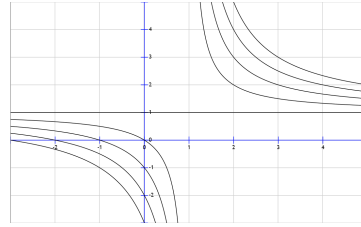
(a)



(b)



(c)



(d)

Figure 2: Non unique factorization region for $n = 2$ (Figure 2a), $n = 3$ (Figure 2b), $n = 4$ (Figure 2c) and $n = 5$ (Figure 2d) sharing nodes.

2. Preliminaries

The mathematical model of a network whose nodes are perfect and its edges can fail is a stochastic graph [Co], [Sh]; i.e. an undirected graph with associated Bernoulli variables to its edges. I assume throughout the paper that the forward slash means “such that”.

Definition 2.1. An undirected graph G is (V, E, ϕ) such that V and E are finite sets whose elements will be called nodes and edges respectively E and ϕ is a function from E to $\{\{a, b\} / a, b \in V\}$ specifying the nodes attached to each edge. We will make an abuse of notation and denote the graph G simply as $G = (V, E)$. Nodes and edges of G will be denoted by $V(G)$ and $E(G)$ respectively.

Definition 2.2. A stochastic graph G is (V, E, ϕ, Φ) such that (V, E, ϕ) is a graph and $\Phi : E \rightarrow \text{Ber}$ is a function which associates a Bernoulli variable to each edge in such a way that these variables are independent.

Definition 2.3. The graph $G' = (V', E', \phi', \Phi')$ is a subgraph of $G = (V, E, \phi, \Phi)$ if $V' \subset V$, $E' \subset E$, $\phi' = \phi|_{E'}$ and $\Phi' = \Phi|_{E'}$.

Each Bernoulli variable is characterized by a parameter p in the $[0, 1]$ closed interval and we can write a stochastic graph as $(G, \{p_e\}_{e \in E})$ where G is an undirected graph and p_e is the parameter of the variable $\Phi(e)$.

Definition 2.4. A state \mathcal{E} of the graph $G = (V, E)$ is a function $\mathcal{E} : E \rightarrow \{0, 1\}$. An edge e will be called operative if $\mathcal{E}(e) = 1$ and will be called non-operative otherwise.

Definition 2.5. A graph G with distinguished vertices $\{1, 2, \dots, n\} \subset V(G)$ is a pair $(G, \{1, 2, \dots, n\})$. We will denote this pair simply by G .

Definition 2.6. Consider a graph G with distinguished vertices $\{1, 2, \dots, n\}$. For each partition \mathcal{A} of $\{1, 2, \dots, n\}$ denote by $G^{\mathcal{A}}$ the graph resulting from the identification of the vertices $\{1, 2, \dots, n\}$ of G by the partition \mathcal{A} . This is illustrated in Figure 4

Figure 3 shows the notation and the diagram for a partition. We define the product $\mathcal{A} \cdot \mathcal{B}$ as the finer partition coarser than \mathcal{A} and \mathcal{B} . Respect to this product, the set of partitions $Part_n$ is a commutative monoid with unit $e = \{\{1\}, \{2\}, \dots, \{n\}\}$.

$$\{\{1, 3\}, \{2, 4\}\} = \widetilde{13} \widetilde{24} = \begin{array}{c} \text{---} 1 \\ \text{---} 2 \\ \text{---} 3 \\ \text{---} 4 \end{array}$$

Figure 3: Notation and diagram for a partition

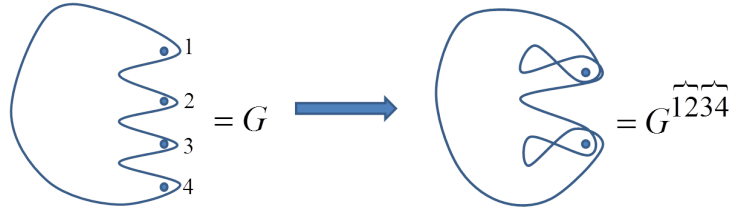


Figure 4: Identificitation of distinguished vertices

3. Extended reliability

For every graph G , we denote by $k(G)$ the number of its connected components. Consider a graph G with distinguished vertices $\{1, 2, \dots, n\}$. We define the equivalence relation \sim_G such that $i \sim_G j$ if and only if i and j belong to the same connected component. The associated partition of G is:

$$\mathcal{P}(G) = \{1, 2, \dots, n\} / \sim_G$$

Definition 3.1. Consider a stochastic graph G . We define the *extended reliability of G* as the polynomial:

$$R_t(G) = \sum_{i=1}^{+\infty} t^{i-1} P(k(G) = i)$$

where $P(k(G) = i)$ is the probability that G has i connected components.

Because the graph G is finite, the maximal number of connected components is the number of vertices $|V(G)|$ and the sum is actually finite. Evaluating the Tutte reliability at $t = 0$ gives the reliability of G . In this sense, the extended reliability defined here is a generalization of the classical reliability.

Definition 3.2. Consider a stochastic graph G with $n \geq 1$ distinguished vertices $\{1, 2, \dots, n\}$ and a partition \mathcal{A} of this set. We define the *extended reliability of G respect to \mathcal{A}* as the polynomial:

$$R_{t,\mathcal{A}}(G) = \sum_{i=1}^{+\infty} t^{i-|\mathcal{A}|} P((k(G) = i) \wedge (\mathcal{P}(G) = \mathcal{A}))$$

where $P((k(G) = i) \wedge (\mathcal{P}(G) = \mathcal{A}))$ is the probability that G has i connected components and its associated partition is \mathcal{A} .

Because of the second condition $\mathcal{P}(G) = \mathcal{A}$ we have that $k(G) \geq |\mathcal{A}|$ hence $R_{t,\mathcal{A}}(G)$ is a polynomial. Given a subgraph $A \subset G$, we will denote by χ_A the indicator function of A such that $\chi_A(e) = 1$ if the edge $e \in A$ and $\chi_A(e) = 0$ otherwise.

Lemma 3.1. *Consider a stochastic graph G such that there are subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$R_t(G) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} R_{t,\mathcal{A}}(G_1) R_{t,\mathcal{B}}(G_2)$$

Proof: For every subgraph $A \subset G$ such that $V(A) = V(G)$ define $A_1 = A \cap G_1$ and denote by C_1 the number of connected components of A_1 not containing any of the distinguished vertices $\{1, 2, \dots, n\}$ with analogous definitions for A_2 and C_2 . Then:

$$k(A_1) = C_1 + |\mathcal{A}|$$

$$k(A_2) = C_2 + |\mathcal{B}|$$

where $\mathcal{P}(A_1) = \mathcal{A}$, $\mathcal{P}(A_2) = \mathcal{B}$ and:

$$k(A) = C_1 + C_2 + |\mathcal{A} \cdot \mathcal{B}| = k(A_1) + k(A_2) + |\mathcal{A} \cdot \mathcal{B}| - |\mathcal{A}| - |\mathcal{B}| \quad (3)$$

Because G_1 and G_2 only share vertices, we have:

$$\begin{aligned}
R_t(G) &= \sum_{\substack{A \subset G \\ V(A)=V(G)}} t^{k(A)-1} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1-\chi_A(e)} \\
&= \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \sum_{\substack{A \subset G \\ V(A)=V(G) \\ \mathcal{P}(A_1)=\mathcal{A}, \mathcal{P}(A_2)=\mathcal{B}}} t^{k(A)-1} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1-\chi_A(e)} \\
&= \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} \left(\sum_{\substack{A_1 \subset G_1 \\ V(A_1)=V(G_1) \\ \mathcal{P}(A_1)=\mathcal{A}}} t^{k(A_1) - |\mathcal{A}|} \prod_{e \in E(G_1)} p_e^{\chi_{A_1}(e)} (1 - p_e)^{1-\chi_{A_1}(e)} \right) \\
&\quad \left(\sum_{\substack{A_2 \subset G_2 \\ V(A_2)=V(G_2) \\ \mathcal{P}(A_2)=\mathcal{B}}} t^{k(A_2) - |\mathcal{B}|} \prod_{e \in E(G_2)} p_e^{\chi_{A_2}(e)} (1 - p_e)^{1-\chi_{A_2}(e)} \right) \\
&= \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} R_{t, \mathcal{A}}(G_1) R_{t, \mathcal{B}}(G_2)
\end{aligned}$$

and we have the result. \square

Lemma 3.2. *Consider a stochastic graph G with $n \geq 1$ distinguished vertices $\{1, 2, \dots, n\}$. Then,*

$$R_t(G^{\mathcal{A}}) = \sum_{\mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} R_{t, \mathcal{B}}(G)$$

Proof: For every subgraph $A \subset G$ such that $V(A) = V(G)$ denote by C the number of connected components of A not containing any of the distinguished vertices $\{1, 2, \dots, n\}$. Then:

$$k(A) = C + |\mathcal{B}|$$

where $\mathcal{P}(A) = \mathcal{B}$ and:

$$k(A^{\mathcal{A}}) = C + |\mathcal{A} \cdot \mathcal{B}| = k(A) + |\mathcal{A} \cdot \mathcal{B}| - |\mathcal{B}| \quad (4)$$

Because $G^{\mathcal{A}}$ only identifies vertices of G we have:

$$\begin{aligned}
R_t(G^{\mathcal{A}}) &= \sum_{\substack{A \subset G \\ V(A)=V(G)}} t^{k(A^{\mathcal{A}})-1} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1-\chi_A(e)} \\
&= \sum_{\mathcal{B} \in \text{Part}_n} \sum_{\substack{A \subset G \\ V(A)=V(G) \\ \mathcal{P}(A)=\mathcal{B}}} t^{k(A^{\mathcal{A}})-1} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1-\chi_A(e)} \\
&= \sum_{\mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} \sum_{\substack{A \subset G \\ V(A)=V(G) \\ \mathcal{P}(A)=\mathcal{B}}} t^{k(A)-|\mathcal{B}|} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1-\chi_A(e)} \\
&= \sum_{\mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}| - 1} R_{t, \mathcal{B}}(G)
\end{aligned}$$

and we have the result. \square

Definition 3.3. Consider $n \geq 1$. We define the *polynomial connectivity matrix*:

$$(t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \in M_{B_n}(\mathbb{Z}[t])$$

where $B_n = |\text{Part}_n|$ is the n -th Bell number.

Lemma 3.3. *The polynomial connectivity matrix is invertible in $M_{B_n}(\mathbb{Q}(t))$ where $\mathbb{Q}(t)$ is the field of rational functions and $B_n = |\text{Part}_n|$ is the n -th Bell number.*

Proof: It follows by the determinant formula 7.1. Nevertheless there is a simple argument in this case: The product of terms on the diagonal is clearly of higher degree than any other product of terms that contributes to the determinant, so the determinant must be nonzero. We still have a third argument: Consider the polynomial connectivity matrix $A(t)$. Because $A(0)$ is the connectivity matrix defined in [BR], [Bu], by the determinant formula proved in [Bu] we have:

$$\det(A(t)) = \pm \prod_{\mathcal{A} \in \text{Part}_n} (|\mathcal{A}| - 1)! \pmod{t} \neq 0$$

\square

Theorem 3.4. • Consider the non empty stochastic graphs G_1 and G_2 .
Then:

$$R_t(G_1 \sqcup G_2) = t R_t(G_1) R_t(G_2)$$

- Consider a stochastic graph G such that there are subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:

$$R_t(G) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} b_{\mathcal{A}\mathcal{B}}(t) R_t(G_1^{\mathcal{A}}) R_t(G_2^{\mathcal{B}})$$

such that:

$$(b_{\mathcal{A}\mathcal{B}}(t))_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \in M(\mathbb{Q}(t))$$

is the inverse of the polynomial connectivity matrix $(t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n}$.

Proof:

- Define $G = G_1 \sqcup G_2$ and for every subgraph $A \subset G_1 \sqcup G_2$ define $A_1 = A \cap G_1$ and $A_2 = A \cap G_2$. Because $k(A) = k(A_1) + k(A_2)$ we have:

$$\begin{aligned} R_t(G) &= \sum_{\substack{A \subset G \\ V(A) = V(G)}} t^{k(A)-1} \prod_{e \in E(G)} p_e^{\chi_A(e)} (1 - p_e)^{1 - \chi_A(e)} \\ &= t \left(\sum_{\substack{A_1 \subset G_1 \\ V(A_1) = V(G_1)}} t^{k(A_1)-1} \prod_{e \in E(G_1)} p_e^{\chi_{A_1}(e)} (1 - p_e)^{1 - \chi_{A_1}(e)} \right) \\ &\quad \left(\sum_{\substack{A_2 \subset G_2 \\ V(A_2) = V(G_2)}} t^{k(A_2)-1} \prod_{e \in E(G_2)} p_e^{\chi_{A_2}(e)} (1 - p_e)^{1 - \chi_{A_2}(e)} \right) \\ &= t R_t(G_1) R_t(G_2) \end{aligned}$$

and we have the first item.

- Define the column vectors:

$$\mathbf{x}_i = (R_{t, \mathcal{A}}(G_i))_{\mathcal{A} \in \text{Part}_n}$$

$$G = \frac{1}{t-1} \left(\begin{aligned} & \text{Graph 1} - \text{Graph 2} \\ & - \text{Graph 3} + t \text{Graph 4} \end{aligned} \right)$$

Figure 5: Extended reliability factorization for two sharing nodes

$$\mathbf{y}_i = (R_t(G_i^A))_{A \in Part_n}$$

where $i = 1, 2$. Consider the polynomial connectivity matrix $A(t)$ and its inverse $B(t)$. Because of Lemmas 3.1 and 3.2 and the fact that $B(t)$ is symmetric we have:

$$R_t(G) = \mathbf{x}_1^t A(t) \mathbf{x}_2 = \mathbf{y}_1^t (B(t)^t A(t) B(t)) \mathbf{y}_2 = \mathbf{y}_1^t B(t) \mathbf{y}_2$$

and we have the second item. □

It is interesting that defining $R_t(\emptyset) = t^{-1}$, the first item is also valid for empty graphs. The case $n = 1$ is clear and reproduces the expected well known factorization respect to an articulation point:

$$R_t(G) = R_t(G_1) R_t(G_2)$$

Let's see how the theorem works for $n = 2$. Ordering the basis $Part_2$ by

$$Part_2 = \{ 12, \widehat{12} \}$$

we get the connectivity matrix:

$$A(t) = \begin{pmatrix} t & 1 \\ 1 & 1 \end{pmatrix}$$

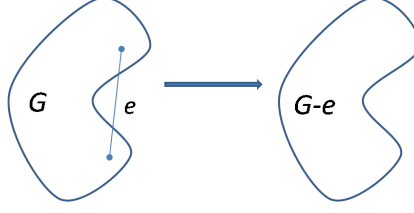


Figure 6: Deletion of the edge e

and its inverse

$$B(t) = \frac{1}{t-1} \begin{pmatrix} 1 & -1 \\ -1 & t \end{pmatrix}$$

hence the factorization reads:

$$\begin{aligned} R_t(G) = & \frac{1}{t-1} \left(R_t(G_1) R_t(G_2) - R_t(G_1) R_t(G_2^{\widehat{12}}) \right. \\ & \left. - R_t(G_1^{\widehat{12}}) R_t(G_2) + t R_t(G_1^{\widehat{12}}) R_t(G_2^{\widehat{12}}) \right) \end{aligned} \quad (5)$$

Figure 5 shows this factorization.

Definition 3.4. Consider an edge $e \in E$ of a graph $G = (V, E, \phi)$ and define the following equivalence relation in V : $a \sim b$ if $a = b$ or $\{a, b\} = \phi(e)$. Consider the surjective canonical function $\pi : V \rightarrow V / \sim$ such that $\pi(a) = [a]_{\sim}$. We define the contraction of an edge e in G as the graph $G \cdot e$ such that $G \cdot e = (V / \sim, E - \{e\}, \bar{\phi})$ where $\bar{\phi}(e) = \{\pi(a), \pi(b)\}$ if $\phi(e) = \{a, b\}$ (see Figure 7).

Definition 3.5. Consider an edge $e \in E$ of the graph $G = (V, E)$. We define the deletion of the edge e of G as the graph (see Figure 6) $G - e = (V, E - \{e\}, \phi)$

The following is the extended reliability simple factorization. The classical case ($t = 0$) was proposed for the first time by Moskowitz [Mo]:

Corollary 3.5. Consider a stochastic graph G and let e be an edge with Bernoulli parameter p . Then:

$$R_t(G) = p R_t(G \cdot e) + (1 - p) R_t(G - e)$$

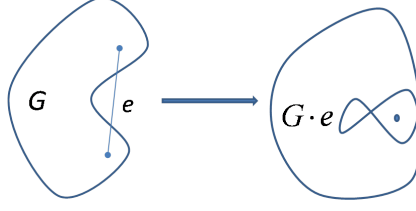


Figure 7: Contraction of the edge e

Proof: Define the stochastic graphs $G_1 = G - e$ and $G_2 = e$ with sharing vertices $\{1, 2\}$. Because $G_1^{12} = G \cdot e$, $R_t(G_1^{12}) = 1$ and $R_t(G_2) = p + t(1 - p)$, by formula (5) we have the result. \square

The above corollary justifies the name of *extended reliability*. In particular, Corollary 3.5 gives the following alternative definition of the extended reliability:

Definition 3.6. Consider a stochastic graph G . The *extended reliability* of G is the polynomial R_t such that:

$$R_t(G) = p_e R_t(G \cdot e) + (1 - p_e) R_t(G - e)$$

where p_e is the Bernoulli parameter of the edge e and:

$$R_t(n \text{ vertices}) = t^{n-1}$$

4. Negami's formula

Recall the definition of the Tutte polynomial of a graph G :

$$T_G(x, y) = \sum_{\substack{A \subseteq G \\ V(A) = V(G)}} (x - 1)^{k(A) - k(G)} (y - 1)^{k(A) + |E(A)| - |V(G)|} \quad (6)$$

Define the *extended reliability polynomial*:

$$R_G(t, p) = R_t((G, p))$$

where (G, p) denotes the stochastic graph whose respective Bernoulli parameters are all equal to p . Then we have:

Lemma 4.1.

$$T_G(x, y) = (x-1)^{1-k(G)}(y-1)^{1-|V(G)|} y^{|E(G)|} R_G \left((x-1)(y-1), \frac{y-1}{y} \right)$$

Proof: Because the edges are associated to identical and independent Bernoulli variables with parameter p we have:

$$P(k(G) = i) = \sum_{\substack{A \subset G \\ V(A)=V(G) \\ k(A)=i}} p^{|E(A)|} (1-p)^{|E(G)|-|E(A)|}$$

Then, the Tutte reliability reads:

$$R_G(t, p) = \sum_{\substack{A \subset G \\ V(A)=V(G)}} t^{k(A)-1} p^{|E(A)|} (1-p)^{|E(G)|-|E(A)|}$$

Taking the change of variables:

$$\begin{aligned} t &= (x-1)(y-1) \\ p &= \frac{y-1}{y} \end{aligned}$$

and comparing with expression (6), we have the result. \square

In particular, we have the Negami's formula [Ne]:

Theorem 4.2. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_G(x, y) = \sum_{A, B \in \text{Part}_n} c_{AB}(x, y) T_{G_1^A}(x, y) T_{G_2^B}(x, y)$$

such that:

$$c_{AB}(x, y) = b_{AB}((x-1)(y-1)) (y-1)^{|A|+|B|-n-1} \quad (7)$$

where $(b_{AB}(t))_{A, B \in \text{Part}_n}$ is the inverse of the polynomial connectivity matrix $(t^{|A \cdot B|-1})_{A, B \in \text{Part}_n}$.

Proof: By hypothesis $k(G) = k(G_1) = k(G_2) = 1$. In particular $k(G_1^{\mathcal{A}}) = 1$ for every partition \mathcal{A} and analogous relation for G_2 . We also have the relations:

$$\begin{aligned} V(G) &= V(G_1) + V(G_2) - n \\ V(G_1^{\mathcal{A}}) &= V(G_1) + |\mathcal{A}| - n \\ V(G_2^{\mathcal{B}}) &= V(G_2) + |\mathcal{B}| - n \\ E(G_1^{\mathcal{A}}) &= E(G_1) \\ E(G_2^{\mathcal{B}}) &= E(G_2) \\ E(G) &= E(G_1) + E(G_2) \end{aligned}$$

By Theorem 3.4 and Lemma 4.1, we have the result. \square

Returning to the previous $n = 2$ case, respect to the basis $\{12, \widehat{12}\}$ we have:

$$(c_{\mathcal{AB}}(x, y)) = \frac{1}{(x-1)(y-1)-1} \begin{pmatrix} y-1 & -1 \\ -1 & x-1 \end{pmatrix}$$

and we have rederived Brylawski's formula [Br]:

$$\begin{aligned} T_G(x, y) &= \frac{1}{(x-1)(y-1)-1} \left((y-1) T_{G_1}(x, y) T_{G_2}(x, y) - T_{G_1}(x, y) T_{G_2^{\widehat{12}}}(x, y) \right. \\ &\quad \left. - T_{G_1^{\widehat{12}}}(x, y) T_{G_2}(x, y) + (x-1) T_{G_1^{\widehat{12}}}(x, y) T_{G_2^{\widehat{12}}}(x, y) \right) \end{aligned} \quad (8)$$

Figure 1 shows this factorization.

5. Specializations I: Extended Reliability

So far we have consider the Tutte polynomial as an element of the polynomial ring $\mathbb{Z}[x, y]$ and the factorization coefficients $c_{\mathcal{AB}}(x, y)$ as elements of the field $\mathbb{Q}(x, y)$. This is the context in which the Tutte general factorization Theorem 4.2 works.

However, because of the Tutte polynomial universality, different branches of mathematics, physics and engineering use different specializations of it and the factorization coefficients $c_{\mathcal{AB}}(x, y)$ could not be defined on these specializations for certain number of sharing nodes. Table 1 show some commonly used specializations.

In general, by definition of the factorization coefficients (7), Tutte general **unique** factorization doesn't exist on the plane curves $y = 1$ (if $n \geq 4$, this is not obvious and will be treated in the next section), $(x-1)(y-1) - \lambda = 0$ and their particular points such that $\lambda = 1, 2, \dots, n-1$.

We first prove the result for extended reliability and then translate it to the Tutte polynomial:

Theorem 5.1. *Consider a stochastic graph G such that there are subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$. Then, for every $t \in \mathbb{C}$ we have:*

$$R_t(G) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} b_{\mathcal{A}\mathcal{B}}(t) R_t(G_1^{\mathcal{A}}) R_t(G_2^{\mathcal{B}})$$

if and only if $B(t) = (b_{\mathcal{A}\mathcal{B}}(t))$ is a solution of the equation $A(t)B(t)A(t) = A(t)$ where $A(t) = (t^{|\mathcal{A} \cdot \mathcal{B}| - 1})$. Moreover, the affine matrix space $\mathcal{M}_{n,t} \subset M_{B_n}(\mathbb{C})$ of solutions of the equation $A(t)B(t)A(t) = A(t)$ has the following dimension:

$$\dim_{\mathbb{C}} \mathcal{M}_{n,t} = \begin{cases} B_n^2 - \left(\sum_{i=1}^t \begin{Bmatrix} n \\ i \end{Bmatrix} \right)^2 & t = 1, 2, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

and the latter case holds if and only if the polynomial connectivity matrix evaluated at t is invertible and $(b_{\mathcal{A}\mathcal{B}}(t))$ is its inverse.

Proof: We divide the proof into three steps:

1. *For each t , the set of functions $\{R_t\}$ is linearly independent:* For every partition $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ of $\{1, 2, \dots, n\}$ define the stochastic graph $\hat{\mathcal{A}}$ with vertices $\{1, 2, \dots, n, a_1, a_2, \dots, a_k\}$ and an edge between i and a_j if $i \in a_j$ with Bernoulli parameter $p = 1$. Its distinguished vertices are $\{1, 2, \dots, n\}$. By direct calculation, for each t we have $R_{t,\mathcal{B}}(\hat{\mathcal{A}}) = \delta_{\mathcal{A}\mathcal{B}}$. In particular, we have the claim.
2. *For each t , $B(t)$ is a factorization matrix if and only if $A(t)B(t)A(t) = A(t)$ where $A(t)$ is the polynomial connectivity matrix evaluated at t :* This is direct calculation. Just as before, define the column vectors:

$$\mathbf{x}_i = (R_{t,\mathcal{A}}(G_i))_{\mathcal{A} \in \text{Part}_n}$$

$$\mathbf{y}_i = (R_t(G_i^A))_{A \in \text{Part}_n}$$

where $i = 1, 2$. Consider the matrix $B(t)$ such that $A(t)B(t)A(t) = A(t)$. Because of Lemmas 3.1 and 3.2 and the fact that $A(t)$ is symmetric we have:

$$R_t(G) = \mathbf{x}_1^t A(t) \mathbf{x}_2 = \mathbf{x}_1^t (A(t)^t B(t) A(t)) \mathbf{x}_2 = \mathbf{y}_1^t B(t) \mathbf{y}_2$$

hence $B(t)$ is a factorization matrix. Because the latter expression is valid for every graph G verifying the hypothesis, it also proves the converse assertion for the set components of the vectors \mathbf{x}_i is linearly independent. Then,

$$\mathcal{M}_{n,t} = \{ B(t) \in M_{B_n}(\mathbb{C}) \text{ such that } A(t)B(t)A(t) = A(t) \}$$

3. *Dimension of the affine space $\mathcal{M}_{n,t}$:* By Lemmas 7.5, 7.4 and Corollary 7.11, there is an invertible matrix $\Lambda \in M_{B_n}(\mathbb{Z})$ such that:

$$\Lambda \cdot A(t) \cdot \Lambda^t = \begin{pmatrix} (t-n+1) \dots (t-1) & 0 & \dots & 0 & 0 \\ 0 & (t-n+2) \dots (t-1) I_{\{n-1\}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (t-1) I_{\{2\}} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

If $t = 1, 2, \dots, n-1$ then all the solutions to the equation $A(t)B(t)A(t) = A(t)$ are:

$$B(t) = \Lambda^t \begin{pmatrix} M_1 & M_2 \\ M_3 & D_k^{-1} \end{pmatrix} \Lambda \in M_{B_n}(\mathbb{C}) \quad (9)$$

such that M_1, M_2, M_3 are arbitrary matrices and D_k is the square $k \times k$ invertible matrix constituted by the first t lower diagonal blocks of $\Lambda \cdot A(t) \cdot \Lambda^t$. In particular:

$$k = \sum_{i=1}^t \left\{ \begin{matrix} n \\ i \end{matrix} \right\}$$

and we have:

$$\dim_{\mathbb{C}} \mathcal{M}_{n,t} = B_n^2 - \left(\sum_{i=1}^t \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \right)^2$$

By the explicit description of the factorization matrices (9), we have that $\dim_{\mathbb{C}} \mathcal{M}_{n,t} = 0$ if and only if $A(t)$ is invertible and $B(t)$ is its inverse.

□

The non unique factorization at $t = 1$, the singularity in formula (5), follows from the fact that $R_1(G) = 1$ for every stochastic graph G . The other non unique factorizations are much more interesting. Table 2 shows some non zero dimensions of the affine spaces $\mathcal{M}_{n,t}$.

6. Specializations II: Tutte polynomial

By Lemma 4.1 and Theorem 4.2, the Tutte polynomial has a new problematic specialization in the line $y = 1$ and their particular points. These are the only remaining specializations to discuss. Through Lemma 4.1 we translate Theorem 5.1 to the Tutte polynomial in the region $y \neq 1$:

Theorem 6.1. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_{G,\lambda}(x, y) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} c_{\mathcal{A}\mathcal{B}}(x, y) T_{G_1^{\mathcal{A}}, \lambda}(x, y) T_{G_2^{\mathcal{B}}, \lambda}(x, y)$$

where the subscript λ denotes the restriction of the Tutte polynomial in the region $(x - 1)(y - 1) - \lambda = 0$, such that:

$$c_{\mathcal{A}\mathcal{B}}(x, y) = b_{\mathcal{A}\mathcal{B}}(\lambda) (y - 1)^{|\mathcal{A}| + |\mathcal{B}| - n - 1} \quad (10)$$

where $(b_{\mathcal{A}\mathcal{B}}(t))_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} \in \mathcal{M}_{n,t}$ is a solution of the equation $A(t)B(t)A(t) = A(t)$ where $A(t) = (t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n}$.

Let's see the specialization to the curve $y = 1$ and their particular points.

Definition 6.1. Consider a connected graph G with n distinguished vertices. We define:

$$T_{G,\mathcal{A}}(x, y) = \sum_{\substack{A \subset G \\ V(A) = V(G) \\ \mathcal{P}(A) = \mathcal{A}}} (x - 1)^{k(A) - |\mathcal{A}|} (y - 1)^{k(A) + |E(A)| - |V(G)|}$$

for every partition $\mathcal{A} \in \text{Part}_n$.

Lemma 6.2. *Consider a connected graph G with n distinguished vertices. Then,*

$$\lim_{\substack{p \rightarrow 0 \\ t/p = x-1}} p^{|\mathcal{A}| - |V(G)|} R_{t,\mathcal{A}}((G, p)) = T_{G,\mathcal{A}}(x, 1)$$

where (G, p) denotes the stochastic graph whose respective Bernoulli parameters are all equal to p .

Proof: Every subgraph $A \subset G$ such that $V(A) = V(G)$ verifies:

$$k(A) + |E(A)| \geq |V(G)|$$

and the equality holds if and only if A is a spanning forest with $k(A)$ trees. Then,

$$T_{G,\mathcal{A}}(x, 1) = \sum_{i=1}^{+\infty} (x-1)^{i-|\mathcal{A}|} S_{i,\mathcal{A}} \quad (11)$$

where $S_{i,\mathcal{A}}$ is the number of spanning forests with i trees and associated partition \mathcal{A} . On the other hand we have:

$$\begin{aligned} R_{t,\mathcal{A}}((G, p)) &= \sum_{i=1}^{+\infty} t^{i-|\mathcal{A}|} P((k(G) = i) \wedge (\mathcal{P}(G) = \mathcal{A})) \\ &= \sum_{i=|\mathcal{A}|}^{|V(G)|-n+|\mathcal{A}|} t^{i-|\mathcal{A}|} (S_{i,\mathcal{A}} p^{|V(G)|-i} + \mathbf{O}(p^{|V(G)|-i+1})) \end{aligned}$$

Then,

$$p^{|\mathcal{A}| - |V(G)|} R_{t,\mathcal{A}}((G, p)) = \sum_{i=|\mathcal{A}|}^{|V(G)|-n+|\mathcal{A}|} (t/p)^{i-|\mathcal{A}|} (S_{i,\mathcal{A}} + \mathbf{O}(p))$$

Taking the limit the result follows. \square

Reasoning exactly as in the previous proof, we have:

$$T_G(x, 1) = \sum_{i=1}^{+\infty} (x-1)^{i-k(G)} S_i$$

where S_i is the number of spanning forests with i trees. This is why $T_G(x, 1)$ is called the *spanning forest number generator*.

For every partition $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ of $\{1, 2, \dots, n\}$ define the graph $\hat{\mathcal{A}}$ with vertices $\{1, 2, \dots, n, a_1, a_2, \dots, a_k\}$ and an edge between i and a_j if $i \in a_j$. Its distinguished vertices are $\{1, 2, \dots, n\}$. Consider the graphs G_1 and G_2 with distinguished vertices $\{1, 2, \dots, n\}$ and define the *join* of them as follows:

$$G_1 * G_2 = (G_1 \times \{0\} \sqcup G_2 \times \{1\}) / \sim$$

such that \sim is the equivalence relation generated by the relation $(i, 0) \sim (i, 1)$ and the distinguished vertices are $[(i, 0)]$.

Definition 6.2. Consider a pair of partitions $\mathcal{A}, \mathcal{B} \in \text{Part}_n$. We say that the pair $(\mathcal{A}, \mathcal{B})$ is connected and simply connected if $\hat{\mathcal{A}} * \hat{\mathcal{B}}$ is a tree.

Lemma 6.3. For every pair of partitions $\mathcal{A}, \mathcal{B} \in \text{Part}_n$ we have:

$$|\mathcal{A}| + |\mathcal{B}| - |\mathcal{A} \cdot \mathcal{B}| \leq n \quad (12)$$

Moreover, if $|\mathcal{A} \cdot \mathcal{B}| = 1$ then the equality holds if and only if $(\mathcal{A}, \mathcal{B})$ is connected and simply connected.

Proof: The result is clear for $n = 1$. Suppose the result holds for $n-1$ and consider partitions $\mathcal{A}, \mathcal{B} \in \text{Part}_n$. These partitions determine equivalence relations $\sim_{\mathcal{A}}$ and $\sim_{\mathcal{B}}$ respectively such that:

$$\mathcal{A} = \{1, 2, \dots, n\} / \sim_{\mathcal{A}}$$

$$\mathcal{B} = \{1, 2, \dots, n\} / \sim_{\mathcal{B}}$$

Denote by the same notation as before the equivalence relations restricted to $\{1, 2, \dots, n-1\}$ and define:

$$\mathcal{A}_{n-1} = \{1, 2, \dots, n-1\} / \sim_{\mathcal{A}}$$

$$\mathcal{B}_{n-1} = \{1, 2, \dots, n-1\} / \sim_{\mathcal{B}}$$

Case 1: If n is only equivalent to itself respect to $\sim_{\mathcal{A}}$ and $\sim_{\mathcal{B}}$, then:

$$|\mathcal{A}| = |\mathcal{A}_{n-1}| + 1$$

$$|\mathcal{B}| = |\mathcal{B}_{n-1}| + 1$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}_{n-1} \cdot \mathcal{B}_{n-1}| + 1$$

and because of the inductive hypothesis the relation follows.

Case 2: If n is only equivalent to itself respect to $\sim_{\mathcal{A}}$ and equivalent to some other element respect to $\sim_{\mathcal{B}}$, then:

$$|\mathcal{A}| = |\mathcal{A}_{n-1}| + 1$$

$$|\mathcal{B}| = |\mathcal{B}_{n-1}|$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}_{n-1} \cdot \mathcal{B}_{n-1}|$$

and because of the inductive hypothesis the relation follows.

Case 3: If n is only equivalent to itself respect to $\sim_{\mathcal{B}}$ and equivalent to some other element respect to $\sim_{\mathcal{A}}$, then:

$$|\mathcal{A}| = |\mathcal{A}_{n-1}|$$

$$|\mathcal{B}| = |\mathcal{B}_{n-1}| + 1$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}_{n-1} \cdot \mathcal{B}_{n-1}|$$

and because of the inductive hypothesis the relation follows.

Case 4: If n is equivalent to some other element respect to $\sim_{\mathcal{A}}$ and $\sim_{\mathcal{B}}$, then:

$$|\mathcal{A}| = |\mathcal{A}_{n-1}|$$

$$|\mathcal{B}| = |\mathcal{B}_{n-1}|$$

$$|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}_{n-1} \cdot \mathcal{B}_{n-1}|$$

and because of the inductive hypothesis the relation follows.

Suppose that $|\mathcal{A} \cdot \mathcal{B}| = 1$. Then $|\mathcal{A}_{n-1} \cdot \mathcal{B}_{n-1}| = 1$. If equality holds in relation 12 then only the second and third case above are possible. In particular we also have the equality:

$$|\mathcal{A}_{n-1}| + |\mathcal{B}_{n-1}| - 1 = n - 1 \tag{13}$$

and by the inductive hypothesis we have that the pair $(\mathcal{A}_{n-1}, \mathcal{B}_{n-1})$ is connected and simply connected. Because only the second and third case are possible we conclude that the pair $(\mathcal{A}, \mathcal{B})$ is connected and simply connected. Conversely, if the pair $(\mathcal{A}, \mathcal{B})$ is connected and simply connected, then the pair $(\mathcal{A}_{n-1}, \mathcal{B}_{n-1})$ is connected and simply connected and only the second and third cases are possible. By the inductive hypothesis we have equality 13 again and because only the second and third cases are possible we have the equality in relation 12. We have proved the Lemma. \square

Lemma 6.4. *Consider a connected graph G with n distinguished vertices. Then,*

$$T_{G,\mathcal{A}}(x, 1) = \sum_{\mathcal{B} \in \text{Part}_n} (x-1)^{|\mathcal{A} \cdot \mathcal{B}|-1} \delta_{n+|\mathcal{A} \cdot \mathcal{B}|-|\mathcal{A}|-|\mathcal{B}|} T_{G,\mathcal{B}}(x, 1)$$

Proof: By Lemma 3.2 we have:

$$\begin{aligned} & p^{1+n-|V(G)|-|\mathcal{A}|} R_t((G^{\mathcal{A}}, p)) \\ &= \sum_{\mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}|-1} p^{1+n-|\mathcal{A}|-|\mathcal{B}|} p^{|\mathcal{B}|-|V(G)|} R_{t,\mathcal{B}}((G, p)) \\ &= \sum_{\mathcal{B} \in \text{Part}_n} (t/p)^{|\mathcal{A} \cdot \mathcal{B}|-1} p^{n+|\mathcal{A} \cdot \mathcal{B}|-|\mathcal{A}|-|\mathcal{B}|} p^{|\mathcal{B}|-|V(G)|} R_{t,\mathcal{B}}((G, p)) \end{aligned}$$

Because $|V(G^{\mathcal{A}})| = |V(G)| - n + |\mathcal{A}|$ and relation (12), by Lemma 6.2 taking the limit $p \rightarrow 0$ such that $t/p = x - 1$ we have the result. \square

Lemma 6.5. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:*

$$T_G(x, 1) = \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} (x-1)^{|\mathcal{A} \cdot \mathcal{B}|-1} \delta_{n+|\mathcal{A} \cdot \mathcal{B}|-|\mathcal{A}|-|\mathcal{B}|} T_{G_1,\mathcal{A}}(x, 1) T_{G_2,\mathcal{B}}(x, 1)$$

Proof: Because $|V(G_1)| + |V(G_2)| - |V(G)| = n$, by Lemma 3.1 we have:

$$\begin{aligned} & p^{1-|V(G)|} R_t((G, p)) \\ &= \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} t^{|\mathcal{A} \cdot \mathcal{B}|-1} p^{1+n-|\mathcal{A}|-|\mathcal{B}|} p^{|\mathcal{A}|-|V(G_1)|} R_{t,\mathcal{A}}((G_1, p)) p^{|\mathcal{B}|-|V(G_2)|} R_{t,\mathcal{B}}((G_2, p)) \\ &= \sum_{\mathcal{A}, \mathcal{B} \in \text{Part}_n} (t/p)^{|\mathcal{A} \cdot \mathcal{B}|-1} p^{n+|\mathcal{A} \cdot \mathcal{B}|-|\mathcal{A}|-|\mathcal{B}|} p^{|\mathcal{A}|-|V(G_1)|} R_{t,\mathcal{A}}((G_1, p)) p^{|\mathcal{B}|-|V(G_2)|} R_{t,\mathcal{B}}((G_2, p)) \end{aligned}$$

Because of relation (12), by Lemma 6.2 taking the limit $p \rightarrow 0$ such that $t/p = x - 1$ we have the result. \square

The following is the factorization Theorem for the specialization $y = 1$ and particular points $(x, 1)$.

Theorem 6.6. *Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G =$*

$G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$ and we have:

$$T_G(x, 1) = \sum_{A, B \in \text{Part}_n} c_{AB}(x, 1) T_{G_1^A}(x, 1) T_{G_2^B}(x, 1)$$

such that $C(x) = (c_{AB}(x, 1))$ is a solution of the equation $A(x)C(x)A(x) = A(x)$ where

$$A(x) = (a_{AB}(x, 1)) = ((x - 1)^{|\mathcal{A} \cdot \mathcal{B}| - 1} \delta_{n + |\mathcal{A} \cdot \mathcal{B}| - |\mathcal{A}| - |\mathcal{B}|})$$

Moreover, the equation $A(x)C(x)A(x) = A(x)$ has a unique solution $C(x)$ if and only if $n \leq 3$.

Remark: If the specialization is on the line $y = 1$, the matrices $A(x)$ and $B(x)$ have polynomial entries in $\mathbb{R}[x]$ and $A(x)C(x)A(x) = A(x)$ is an equation of polynomial matrices. If the specialization is on a particular point $(x, 1)$, the matrices $A(x)$ and $B(x)$ have real entries and $A(x)C(x)A(x) = A(x)$ is an equation of real matrices.

Proof: This is a direct calculation. Just as before, define the column vectors:

$$\begin{aligned} \mathbf{x}_i &= (T_{G_i, A}(x, 1))_{A \in \text{Part}_n} \\ \mathbf{y}_i &= (T_{G_i^A}(x, 1))_{A \in \text{Part}_n} \end{aligned}$$

where $i = 1, 2$. Consider the matrix C such that $ACA = A$. Because of Lemmas 6.4 and 6.5 and the fact that A is symmetric we have:

$$T_G(x, 1) = \mathbf{x}_1^t A \mathbf{x}_2 = \mathbf{x}_1^t (ACA) \mathbf{x}_2 = \mathbf{y}_1^t C \mathbf{y}_2$$

hence C is a factorization matrix.

The equation $ACA = A$ has a unique solution C if and only A is invertible. By direct calculation, the matrix A is invertible for $n = 1, 2, 3$ sharing nodes. It rest to show that A is non invertible if $n \geq 4$. Consider an ordering of Part_n such that $|\mathcal{A}_i| < |\mathcal{A}_j|$ implies $i < j$. Taking the unit entry of the left inferior corner as a pivot, Gauss elimination gives a block with zero and one entries of dimension:

$$\begin{Bmatrix} n \\ 2 \end{Bmatrix} \times \begin{Bmatrix} n \\ n - 1 \end{Bmatrix} \quad (14)$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ are the Stirling numbers. Because of the following relations:

$$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1$$

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2} = \frac{n(n-1)}{2}$$

we have that:

$$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} > \left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$$

for $n \geq 4$. In particular, for $n \geq 4$, Gauss elimination on the block (14) gives a zero row and the proof is complete. \square

In particular, by Lemma 6.3 we have $a_{\mathcal{A}\mathcal{B}}(1,1) = 1$ if the pair $(\mathcal{A}, \mathcal{B})$ is connected and simply connected and $a_{\mathcal{A}\mathcal{B}}(1,1) = 0$ otherwise. We call the matrix $(a_{\mathcal{A}\mathcal{B}}(1,1))$ the simply connectivity matrix. The following Corollary is illustrated in Figure 2.

Corollary 6.7. *There is no Tutte polynomial unique general factorization for n sharing nodes only in the specialization to the curve $(x-1)(y-1)-\lambda=0$ and their particular points such that $\lambda=1, 2, \dots, n-1$ and the curve $y=1$ and their particular points if $n \geq 4$; i.e. The spanning forest number generator, the q -state Potts model and their particularizations are the only models with no unique general factorization for $n \geq q+1$ and $n \geq 4$ respectively.*

7. Determinant formula

7.1. Introduction

This section is devoted to the proof of the determinant formula:

Theorem 7.1. *Denote by $Part_n$ the set of partitions of n elements and $|\mathcal{A}|$ the number of classes in the partition \mathcal{A} . Then:*

$$\det \left((t^{|\mathcal{A} \cdot \mathcal{B}| - 1})_{\mathcal{A}, \mathcal{B} \in Part_n} \right) = \prod_{\mathcal{A} \in Part_n} \prod_{\lambda=1}^{|\mathcal{A}|-1} (t - \lambda)$$

where the product $\mathcal{A} \cdot \mathcal{B}$ is the finer partition coarser than \mathcal{A} and \mathcal{B} .

The mathematical formalisms in the proof are essentially modern algebra [BMa] and combinatorics of network reliability [Co],[Sh]. The proof will comprise four steps:

1. Coherent order: We define an appropriate total order on the partitions set compatible with its natural partial order.
2. Gauss elimination: Through elementary row operations, for each n we develop a Gauss elimination method such that the connectivity matrix becomes lower triangular.
3. Connectivity polynomials and extended reliability: Identify the resulting diagonal elements, called connectivity polynomials, with the respective higher order term coefficient of certain extended reliability polynomials.
4. Connectivity numbers calculation: Under this identification, calculate the diagonal elements.

7.2. Step 1: Coherent order

Definition 7.1. An ordering of the basis $Part_n$ will be called coherent if $\mathcal{A}_i < \mathcal{A}_j$ implies $i < j$.

We argue that a coherent ordering always exist in the following way: Consider the Hasse diagram (partial ordering diagram) of connectivity states in $Part_n$. Because conjugated states necessary belong to the same level of the Hasse diagram, we can order $Part_n$ in the following way: We order some conjugation class \mathcal{O}_i of the first level, then we order some other conjugation class \mathcal{O}_j of the same level and we continue the process until we have order all the conjugation classes of the first level. After that, we order the second level in the same way as we did in the first and so on until we have order all the partitions. The previous argument is formalized in the next lemma:

Lemma 7.2. *The partial ordering on $Part_n$ induces a partial ordering on the conjugation classes.*

Proof: Define the following partial order on the conjugation classes: $\mathcal{O}_i \preceq \mathcal{O}_j$ if there is $\mathcal{A}_i \in \mathcal{O}_i$ and $\mathcal{A}_j \in \mathcal{O}_j$ such that $\mathcal{A}_i \preceq \mathcal{A}_j$. This partial order relation is well defined because of the following fact: Suppose there are $\mathcal{A}_i, \mathcal{A}'_i \in \mathcal{O}_i$ and $\mathcal{A}_j, \mathcal{A}'_j \in \mathcal{O}_j$ such that $\mathcal{A}_i \preceq \mathcal{A}_j$ and $\mathcal{A}'_i > \mathcal{A}'_j$. There are permutations $\sigma, \eta \in S_n$ such that $\mathcal{A}'_i = \sigma(\mathcal{A}_i)$ and $\mathcal{A}'_j = \eta(\mathcal{A}_j)$ and we have:

$$\mathcal{A}_i \preceq \mathcal{A}_j = \eta^{-1}(\mathcal{A}'_j) < \eta^{-1}(\mathcal{A}'_i) = \eta^{-1}\sigma(\mathcal{A}_i)$$

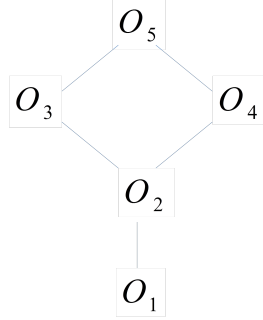


Figure 8: Hasse diagram of conjugation classes $Part_4$

which is absurd for $\eta^{-1}\sigma \in S_n$ and there is no partition with the above property. \square

As an example consider the $n = 4$ case. The conjugation classes are:

$$\begin{aligned}
\mathcal{O}_1 &= \{ 1234 \} \\
\mathcal{O}_2 &= \{ 12 \overbrace{34}, 13 \overbrace{24}, 23 \overbrace{14}, 1 \overbrace{23} 4, \overbrace{13} 24, \overbrace{12} 34 \} \\
\mathcal{O}_3 &= \{ \overbrace{14} \overbrace{23}, \overbrace{13} \overbrace{24}, \overbrace{12} \overbrace{34} \} \\
\mathcal{O}_4 &= \{ 1 \overbrace{234}, 2 \overbrace{134}, 3 \overbrace{124}, \overbrace{123} 4 \} \\
\mathcal{O}_5 &= \{ \overbrace{1234} \}
\end{aligned}$$

Figure 8 shows the Hasse diagram of the induced partial ordering on the conjugation classes of $Part_4$ described in the previous lemma. Is clear then that the following is a coherent order of $Part_4$ (we use the standard linear algebra abuse of set notation concerning ordered basis):

$$Part_4 = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5$$

7.3. Step two: Gauss elimination

We define the product $\mathcal{A} \cdot \mathcal{B}$ as the finer partition coarser than \mathcal{A} and \mathcal{B} . Respect to this product, the set of partitions $Part_n$ is a commutative monoid with unit $e = \{\{1\}, \{2\}, \dots, \{n\}\}$. Then, the \mathbb{Q} -vector space $\mathbb{Q}\langle Part_n \rangle$ generated by the basis $Part_n$ is a commutative associative algebra with unit e

and product:

$$\left(\sum_i a_i \mathcal{A}_i \right) \cdot \left(\sum_j b_j \mathcal{B}_j \right) = \sum_{i,j} a_i b_j (\mathcal{A}_i \cdot \mathcal{B}_j)$$

We denote by $\langle n \rangle$ the trivial partition $\{\{1, 2, \dots, n\}\}$. See that $\mathcal{A} \cdot \langle n \rangle = \langle n \rangle$ for every partition \mathcal{A} .

Because the partition symmetries are encoded in the partition's algebra $\mathbb{Q}\langle Part_n \rangle$, to develop a systematic Gauss elimination on the polynomial connectivity matrix we encode elementary row operations as non trivial identities in this algebra.

Consider the linear operator π on $\mathbb{Q}\langle Part_n \rangle$ such that $\pi(\langle n \rangle) = \langle n \rangle$ and:

$$\pi(\mathcal{A}) = \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B})$$

for every partition \mathcal{A} distinct from $\langle n \rangle$.

Lemma 7.3. *Consider a partition \mathcal{A} . The element $\pi(\mathcal{A})$ satisfies the following properties:*

1. $\mathcal{B} \cdot \pi(\mathcal{A}) = 0 \ \forall \mathcal{B} / \mathcal{B} \not\leq \mathcal{A}$
2. $\mathcal{C} \cdot \pi(\mathcal{A}) = \pi(\mathcal{A}) \ \forall \mathcal{C} / \mathcal{C} \leq \mathcal{A}$
3. $\pi(\sigma(\mathcal{A})) = \sigma(\pi(\mathcal{A})) \ \forall \sigma \in S_n$
In particular, $\mathcal{A} \cdot \pi(\mathcal{A}) = \pi(\mathcal{A})$ and $\mathcal{B} \cdot \pi(\mathcal{A}) = 0$ for every partition \mathcal{B} distinct and conjugated to \mathcal{A} .
4. $\pi(\mathcal{A}) \cdot \pi(\mathcal{B}) = \begin{cases} \pi(\mathcal{A}) & \mathcal{A} = \mathcal{B} \\ 0 & \mathcal{A} \neq \mathcal{B} \end{cases}$
5. $e = \sum_{\mathcal{A} \in Part_n} \pi(\mathcal{A})$

Proof: If $\mathcal{A} = \langle n \rangle$ the result is trivially verified. Consider $\mathcal{A} \neq \langle n \rangle$:

1. The algebra $\mathbb{Q}\langle Part_n \rangle$ is commutative and $Part_n$ is a basis of idempotents; i.e. $\mathcal{A}^2 = \mathcal{A}$ for every partition \mathcal{A} . In particular, there is a factor of $\mathcal{B} \cdot \pi(\mathcal{A})$ that is zero:

$$\mathcal{B} \cdot (\mathcal{A} - \mathcal{A} \cdot \mathcal{B}) = \mathcal{B} \cdot \mathcal{A} - \mathcal{B}^2 \cdot \mathcal{A} = 0$$

2. Because of the fact that $\mathcal{C} \cdot \mathcal{A} = \mathcal{A}$ if $\mathcal{C} \leq \mathcal{A}$, every factor of $\pi(\mathcal{A})$ remains the same after multiplying by \mathcal{C} :

$$\mathcal{C} \cdot (\mathcal{A} - \mathcal{A} \cdot \mathcal{B}) = \mathcal{C} \cdot \mathcal{A} - \mathcal{C} \cdot \mathcal{A} \cdot \mathcal{B} = \mathcal{A} - \mathcal{A} \cdot \mathcal{B}$$

3.

$$\begin{aligned}
\sigma(\pi(\mathcal{A})) &= \sigma \left(\prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B}) \right) \\
&= \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\sigma(\mathcal{A}) - \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})) \\
&= \prod_{\sigma(\mathcal{B}) / \sigma(\mathcal{B}) \not\leq \sigma(\mathcal{A})} (\sigma(\mathcal{A}) - \sigma(\mathcal{A}) \cdot \sigma(\mathcal{B})) \\
&= \pi(\sigma(\mathcal{A}))
\end{aligned}$$

where we have used in the last identity the fact that $\sigma \in S_n$ is invertible.

4. We separate the proof in cases:

- If $\mathcal{B} = \mathcal{A} = \langle n \rangle$ the identity is trivially verified.
- If $\mathcal{B} = \langle n \rangle$ and $\mathcal{A} \neq \langle n \rangle$ then $\pi(\mathcal{A}) \cdot \pi(\mathcal{B}) = \pi(\mathcal{A}) \cdot \langle n \rangle = 0$ by the first item.
- Suppose $\mathcal{B} \neq \langle n \rangle$ and $\mathcal{A} \neq \langle n \rangle$. If $\mathcal{B} \not\leq \mathcal{A}$ then:

$$\pi(\mathcal{A}) \cdot \pi(\mathcal{B}) = \pi(\mathcal{A}) \cdot \mathcal{B} \cdot \left(\prod_{\mathcal{C} / \mathcal{C} \not\leq \mathcal{B}} (e - \mathcal{C}) \right) = 0$$

by the first item. Because every partition is idempotent we have:

$$\begin{aligned}
\pi(\mathcal{A})^2 &= \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B})^2 = \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A}^2 - 2\mathcal{A}^2 \cdot \mathcal{B} + \mathcal{A}^2 \cdot \mathcal{B}^2) \\
&= \prod_{\mathcal{B} / \mathcal{B} \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \mathcal{B}) = \pi(\mathcal{A})
\end{aligned}$$

If $\mathcal{B} \leq \mathcal{A}$, because every partition is idempotent, a similar argument as before gives:

$$\begin{aligned}
\pi(\mathcal{A}) \cdot \pi(\mathcal{B}) &= \mathcal{A} \cdot \left(\prod_{\mathcal{C} / \mathcal{C} \not\leq \mathcal{B}} (e - \mathcal{C}) \right) \\
&= \mathcal{A} \cdot (e - \mathcal{A}) \cdot \left(\prod_{\mathcal{C} / \mathcal{C} \not\leq \mathcal{B}} (e - \mathcal{C}) \right) = 0
\end{aligned}$$

5. The set $\{\pi(\mathcal{A})\}$ is linearly independent for $\sum_{\mathcal{A} \in \text{Part}_n} \lambda_{\mathcal{A}} \pi(\mathcal{A}) = 0$ imply:

$$\lambda_{\mathcal{B}} \pi(\mathcal{B}) = \pi(\mathcal{B}) \cdot \sum_{\mathcal{A} \in \text{Part}_n} \lambda_{\mathcal{A}} \pi(\mathcal{A}) = 0$$

hence $\lambda_{\mathcal{B}} = 0$ for $\pi(\mathcal{B}) \neq 0$ for every partition \mathcal{B} . Because there are as much elements of $\{\pi(\mathcal{A})\}$ as partitions, this set is actually a basis of $\mathbb{Q}\langle \text{Part}_n \rangle$. In particular there are coefficients $c_{\mathcal{A}}$ such that:

$$e = \sum_{\mathcal{A} \in \text{Part}_n} c_{\mathcal{A}} \pi(\mathcal{A})$$

Multiplying the above expression by $\pi(\mathcal{B})$ we have $\pi(\mathcal{B}) = c_{\mathcal{B}} \pi(\mathcal{B})$ hence $c_{\mathcal{B}} = 1$ for every partition \mathcal{B} .

□

Observation 7.1. The fourth and fifth item of the above Lemma imply that $\{\pi(\mathcal{A})\}$ is a complete set of central idempotents. In particular, the algebra $\mathbb{Q}\langle \text{Part}_n \rangle$ is isomorphic to \mathbb{Q}^{B_n} where $B_n = |\text{Part}_n|$ is the n -th Bell number.

Definition 7.2. For each partition \mathcal{A} we define its *connectivity polynomial* $\alpha_{\mathcal{A}}(t)$ such that:

$$\alpha_{\mathcal{A}}(t) = \sum_{\mathcal{B} \in \text{Part}_n} \lambda_{\mathcal{A}\mathcal{B}} t^{|\mathcal{B}|-1}$$

where:

$$\pi(\mathcal{A}) = \sum_{\mathcal{B} \in \text{Part}_n} \lambda_{\mathcal{A}\mathcal{B}} \mathcal{B}$$

Lemma 7.4. 1. For every partition \mathcal{A} the connectivity polynomial $\alpha_{\mathcal{A}}(t)$ is monic with degree $|\mathcal{A}| - 1$.

2. $t^{n-1} = \sum_{\mathcal{A} \in \text{Part}_n} \alpha_{\mathcal{A}}(t)$

3. For every partition \mathcal{A} the connectivity polynomial is invariant under conjugation: $\alpha_{\mathcal{A}}(t) = \alpha_{\sigma(\mathcal{A})}(t)$ for every permutation $\sigma \in S_n$. In particular, the connectivity polynomial is defined on the partition classes \mathcal{O}_i : $\alpha_{\mathcal{O}_i}(t) = \alpha_{\mathcal{A}}(t)$ such that $\mathcal{A} \in \mathcal{O}_i$.

4. Consider a coherent order on the basis Part_n and define the matrix $\Lambda = [\pi]_{\text{Part}_n}^t$; i.e. $\Lambda = (\lambda_{\mathcal{A}\mathcal{B}})_{\mathcal{A}, \mathcal{B} \in \text{Part}_n}$ such that:

$$\pi(\mathcal{A}) = \sum_{\mathcal{B} \in \text{Part}_n} \lambda_{\mathcal{A}\mathcal{B}} \mathcal{B}$$

Then, Λ is an upper triangular matrix with its diagonal entries equal to one:

$$\Lambda = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

In particular $\det(\Lambda) = 1$.

Proof:

1. By definition of π , for every partition \mathcal{A} we have:

$$\pi(\mathcal{A}) = \mathcal{A} + \sum_{\substack{\mathcal{B} \in Part_n \\ |\mathcal{B}| < |\mathcal{A}|}} \lambda_{\mathcal{A}\mathcal{B}} \mathcal{B} \quad (15)$$

In particular, the polynomial $\alpha_{\mathcal{A}}(t)$ is monic with degree $|\mathcal{A}| - 1$:

$$\alpha_{\mathcal{A}}(t) = t^{|\mathcal{A}|-1} + \sum_{\substack{\mathcal{B} \in Part_n \\ |\mathcal{B}| < |\mathcal{A}|}} \lambda_{\mathcal{A}\mathcal{B}} t^{|\mathcal{B}|-1}$$

2. Define the linear transformation $T : \mathbb{Q}\langle Part_n \rangle \rightarrow \mathbb{Q}[t]$ such that $T(\mathcal{A}) = t^{|\mathcal{A}|-1}$. By the fifth item of Lemma 7.3 we have:

$$e = \sum_{\mathcal{A} \in Part_n} \pi(\mathcal{A})$$

Applying the linear map T to both sides of the last identity we have the result.

3. Follows by the third item of Lemma 7.3.
4. Follows by equation (15) and the coherent order definition 7.1.

□

Observe that, once we have ordered the basis $Part_n$ coherently, the operator π describes the Gauss elimination we were looking for. This is the content of the following Lemma:

Lemma 7.5. *Consider a coherent order on the basis $Part_n$ and the induced order on the partition classes $\{\mathcal{O}_1, \dots, \mathcal{O}_k\}$. Then:*

$$\Lambda \cdot A(t) \cdot \Lambda^t = \begin{pmatrix} \alpha_{\mathcal{O}_1}(t) I_{|\mathcal{O}_1|} & 0 & \dots & 0 \\ 0 & \alpha_{\mathcal{O}_2}(t) I_{|\mathcal{O}_2|} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{\mathcal{O}_k}(t) I_{|\mathcal{O}_k|} \end{pmatrix}$$

is a diagonal matrix where $\Lambda = [\pi]_{Part_n}^t$ (see Lemma 7.4).

Proof: Denote by $(a'_{ij}(t)) = \Lambda \cdot A(t) \cdot \Lambda^t$. By definition, $a_{ij}(t) = t^{|\mathcal{A}_i \cdot \mathcal{A}_j| - 1}$ and we can trivially write this expression through the structure constants c_{ij}^k :

$$a_{ij}(t) = \sum_k c_{ij}^k t^{|\mathcal{A}_k| - 1}$$

such that:

$$\mathcal{A}_i \cdot \mathcal{A}_j = \sum_k c_{ij}^k \mathcal{A}_k$$

Then:

$$a'_{ij}(t) = \sum_{l,m} \lambda_{il} a_{lm}(t) \lambda_{jm} = \sum_{k,l,m} \lambda_{il} \lambda_{jm} c_{lm}^k t^{|\mathcal{A}_k| - 1} = \sum_k d_{ij}^k t^{|\mathcal{A}_k| - 1}$$

where $d_{ij}^k = \sum_{l,m} \lambda_{il} \lambda_{jm} c_{lm}^k$ are the following structure constants:

$$\begin{aligned} \pi(\mathcal{A}_i) \cdot \pi(\mathcal{A}_j) &= \left(\sum_l \lambda_{il} \mathcal{A}_l \right) \cdot \left(\sum_m \lambda_{jm} \mathcal{A}_m \right) \\ &= \sum_{l,m} \lambda_{il} \lambda_{jm} (\mathcal{A}_l \cdot \mathcal{A}_m) \\ &= \sum_{k,l,m} \lambda_{il} \lambda_{jm} c_{lm}^k \mathcal{A}_k = \sum_k d_{ij}^k \mathcal{A}_k \end{aligned}$$

By the fourth item of Lemma 7.3, we have the following identity:

$$\sum_{l,m} \lambda_{il} \lambda_{jm} c_{lm}^k = \delta_{ij} \lambda_{ik} \quad (16)$$

By the connectivity polynomial definition 7.2 we have the result. \square

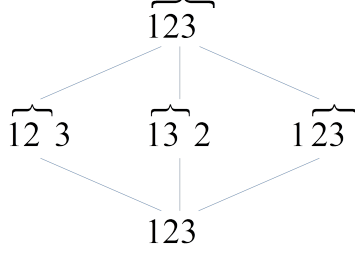


Figure 9: Hasse diagram of $Part_3$

Proposition 7.6. *The determinant of the polynomial connectivity matrix $A(t)$ is:*

$$\det(A(t)) = \prod_{\mathcal{A} \in Part_n} \alpha_{\mathcal{A}}(t)$$

Proof: Choose a coherent order in the basis $Part_n$ and consider the polynomial connectivity matrix $A(t)$ and the matrix Λ respect to this order. By Lemma 7.4 we have $\det(\Lambda) = 1$ hence by Lemma 7.5 we have:

$$\det(A(t)) = \det(\Lambda \cdot A(t) \cdot \Lambda^t) = \prod_{i=1}^k \alpha_{\mathcal{O}_i}(t)^{|\mathcal{O}_i|} = \prod_{\mathcal{A} \in Part_n} \alpha_{\mathcal{A}}(t)$$

and the proof is complete. \square

Observation 7.2. Note that the expressions of Lemma 7.5 and Proposition 7.6 are independent of any ordering in the basis $Part_n$. However, the coherent order makes evident that $\det(\Lambda) = 1$ and the rank of $A(t)$ as a function of t .

As an example consider the $n = 3$ case. The Hasse diagram of $Part_3$ is shown in Figure 9 and is clear that a coherent order on the basis $Part_3$ is

$$Part_3 = \{123, 1 \overbrace{23}, \overbrace{13} 2, \overbrace{12} 3, \overbrace{123}\}$$

The resulting polynomial connectivity matrix is:

$$A(t) = \begin{pmatrix} t^2 & t & t & t & 1 \\ t & t & 1 & 1 & 1 \\ t & 1 & t & 1 & 1 \\ t & 1 & 1 & t & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The linear operator π reads as follows:

$$\begin{aligned}
\pi(\overbrace{123}) &= \overbrace{123} - \overbrace{12} \overbrace{3} - \overbrace{13} \overbrace{2} - 1 \overbrace{23} + 2 \cdot \overbrace{123} \\
\pi(\overbrace{12} \overbrace{3}) &= \overbrace{12} \overbrace{3} - \overbrace{123} \\
\pi(\overbrace{13} \overbrace{2}) &= \overbrace{13} \overbrace{2} - \overbrace{123} \\
\pi(1 \overbrace{23}) &= 1 \overbrace{23} - \overbrace{123} \\
\pi(\overbrace{123}) &= \overbrace{123}
\end{aligned}$$

and the matrix $\Lambda = [\pi]_{Part_n}^t$ (see Lemma 7.4) is:

$$\Lambda = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & 0 & 0 & -1 \\ & & 1 & 0 & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}$$

The matrix Λ expresses the elementary row and column operations on the polynomial connectivity matrix $A(t)$ needed to write it in diagonal form:

$$\Lambda \cdot A(t) \cdot \Lambda^t = \begin{pmatrix} (t-2)(t-1) & & & & \\ & t-1 & & & \\ & & t-1 & & \\ & & & t-1 & \\ & & & & 1 \end{pmatrix} \quad (17)$$

Observe that the diagonal elements of $\Lambda \cdot A(t) \cdot \Lambda^t$ are the respective connectivity polynomials $\alpha_{\mathcal{A}}(t)$. Finally, the determinant of $A(t)$ is:

$$\det(A(t)) = (t-2)(t-1)^4$$

7.4. Step three: Connectivity polynomials and extended reliability

Although the connectivity matrix was motivated from a reliability problem, it is very interesting that the connectivity matrix determinant turns out to be a reliability calculation; i.e. The combinatorics of the connectivity polynomials is encoded in the combinatorics of an extended reliability polynomial.

Definition 7.3. Denote by K_n the graph with n vertices and one edge joining every pair of nodes. Denote by $K_n^{\mathcal{A}}$ the resulting graph from the identification of the vertices $\{1, 2, \dots, n\}$ in K_n by the classes of the partition \mathcal{A} .

Lemma 7.7. Consider a partition \mathcal{A} . Then,

$$R_t(K_n^{\mathcal{A}})(p) = \alpha_{\mathcal{A}}(t) (-p)^g + \dots$$

where $\alpha_{\mathcal{A}}(t)$ is the connectivity polynomial of the partition \mathcal{A} and the dots denote terms of lower degree in p .

Proof: Consider $\mathcal{A} = \langle n \rangle$. Every edge of $K_n^{\langle n \rangle}$ is irrelevant hence $R_t(K_n^{\langle n \rangle}) = 1$ and the result follows because $\alpha_{\langle n \rangle}(t) = 1$. Now, consider a non trivial partition $\mathcal{A} \neq \langle n \rangle$. We claim that:

$$\pi(\mathcal{A}) = \mathcal{A} \cdot \left(\prod_{\tau \text{ minimal} / \tau \not\leq \mathcal{A}} (e - \tau) \right)$$

In effect, consider a partition \mathcal{B} such that $\mathcal{B} \not\leq \mathcal{A}$. There is a minimal partition τ such that $\tau < \mathcal{B}$ and $\tau \not\leq \mathcal{A}$. Because $\mathcal{A} < \mathcal{A} \cdot \mathcal{B}$ and $\tau < \mathcal{A} \cdot \mathcal{B}$ we have that $\mathcal{A} \cdot \tau < \mathcal{A} \cdot \mathcal{B}$ hence:

$$\mathcal{A} \cdot \mathcal{B} \cdot (\mathcal{A} - \mathcal{A} \cdot \tau) = \mathcal{A} \cdot \mathcal{B} - \mathcal{A} \cdot \mathcal{B} = 0$$

In particular, this implies that:

$$\mathcal{A} \cdot \mathcal{B} \cdot \left(\prod_{\tau \text{ minimal} / \tau \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \tau) \right) = 0$$

and we conclude that considering only the minimal partitions not finer than \mathcal{A} is enough for the definition of $\pi(\mathcal{A})$:

$$\pi(\mathcal{A}) = \prod_{\tau \text{ minimal} / \tau \not\leq \mathcal{A}} (\mathcal{A} - \mathcal{A} \cdot \tau) = \mathcal{A} \cdot \left(\prod_{\tau \text{ minimal} / \tau \not\leq \mathcal{A}} (e - \tau) \right)$$

which proves the claim. In particular we have the following expression:

$$\pi(\mathcal{A}) = \sum_{F \subset \{\tau \text{ minimal} / \tau \not\leq \mathcal{A}\}} (-1)^{|F|} \left(\mathcal{A} \cdot \prod_{\tau \in F} \tau \right)$$

hence the connectivity polynomial reads:

$$\begin{aligned}\alpha_{\mathcal{A}}(t) &= \sum_{F \subset \{\tau \text{ minimal} / \tau \not\subset \mathcal{A}\}} (-1)^{|F|} t^{|\mathcal{A} \cdot \prod_{\tau \in F} \tau| - 1} \\ &= \sum_{i=1}^{|\mathcal{A}|} (C_0^i - C_1^i + C_2^i - C_3^i + \dots) t^{i-1}\end{aligned}$$

where C_j^i is the number of subsets F with cardinality j of the set of transpositions $\{\tau \text{ minimal} / \tau \not\subset \mathcal{A}\}$ such that:

$$|\mathcal{A} \cdot \prod_{\tau \in F} \tau| = i$$

Identifying the minimal partition (ij) with the edge joining the nodes i and j of the graph K_n , it is clear that C_j^i is the pathsets number (operational states number) of $K_n^{\mathcal{A}}$ with just j operational edges and i connected components. This way we have:

$$\begin{aligned}R_t(K_n^{\mathcal{A}}) &= \sum_{i=1}^{|\mathcal{A}|} (C_0^i (1-p)^g + C_1^i p(1-p)^{g-1} + C_2^i p^2(1-p)^{g-2} + \dots) t^{i-1} \\ &= (-p)^g \sum_{i=1}^{|\mathcal{A}|} (C_0^i - C_1^i + C_2^i - C_3^i + \dots) t^{i-1} + \dots \\ &= \alpha_{\mathcal{A}}(t) (-p)^g + \dots\end{aligned}$$

where the dots denote terms of lower degree in p . □

Let's return to the $n = 3$ example. We have:

$$R_t(K_3) = (t-2)(t-1)(-p)^3 + \dots$$

$$R_t(K_3^{\widehat{123}}) = (t-1)(-p)^2 + \dots$$

and analogous expressions for $K_3^{\widehat{132}}$ and $K_3^{\widehat{12\bar{3}}}$ where the dots denote terms of lower degree in p . These coefficients reproduce the diagonal elements of matrix (17) previously calculated.

7.5. Step four: Connectivity polynomials calculation

Let G be a graph and consider its extended reliability polynomial $R_t(G)$. Denote by $mgr(R_t(G))$ the term of $R_t(G)$ whose degree in p equals the edge number of G . Observe that if G has an irrelevant edge, then $mgr(R_t(G)) = 0$ and in case $mgr(R_t(G))$ is non zero, then this term equals the highest degree term of the polynomial in p . The following trick will be extremely useful in the following calculations.

Lemma 7.8. *Consider a graph G with k edges between a pair of distinct vertices i and j . Consider the resulting graph \tilde{G} by deleting $k - 1$ edges between the vertices i and j . Then,*

$$mgr(R_t(G)) = (-p)^{k-1} mgr(R_t(\tilde{G}))$$

Proof: The result is clear for $k = 1$. Suppose there are $k > 1$ edges between the vertices i and j of G and that the result holds for an amount less than or equal to $k - 1$ of them. Consider an edge a between the nodes i and j . Simple factorization on the edge a gives

$$R_t(G) = p R_t(G \cdot a) + (1 - p) R_t(G - a)$$

where $G \cdot a$ is the resulting graph by the contraction of a and $G - a$ is the resulting graph by deleting a . Observe that the edge number of $G \cdot a$ and $G - a$ is the edge number of G minus one and because $k > 1$, $G \cdot a$ has irrelevant edges. This way,

$$mgr(R_t(G)) = (-p) mgr(R_t(G - a))$$

By the inductive hypothesis, we get the result. \square

Lemma 7.9. *The extended reliability polynomial of K_n is*

$$R_t(K_n) = \left(\prod_{\lambda=1}^{n-1} (t - \lambda) \right) (-p)^g + \dots$$

where the highest degree g equals the edge number of K_n :

$$g = \binom{n}{2}$$

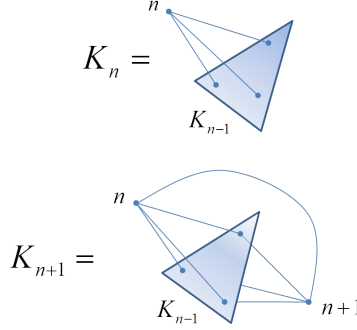


Figure 10: Relation between the graphs K_{n+1} , K_n and K_{n-1}

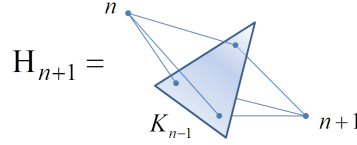


Figure 11: The graph H_{n+1}

Proof: In this proof we will make an abuse of notation identifying the extended reliability polynomial with its graph. We claim that:

$$mgr(K_{n+1}) = (t - n) mgr(K_n) (-p)^n$$

Because $K_2 = (1 - t)p + t$ and $K_1 = 1$ we have the claim for the $n = 1$ case. Suppose the claim is true for every natural number less than or equal to n . Figure 10 shows the relation between the graphs K_{n+1} , K_n and K_{n-1} . By simple factorization on the edge joining the vertices n and $n + 1$ of the graph K_{n+1} and the above Lemma we have:

$$K_{n+1} = p \left((-p)^{n-1} K_n + \dots \right) + (1 - p) H_{n+1} \quad (18)$$

where the dots denote terms whose degree is less than the edge number of K_{n+1} and the graph H_{n+1} results from deleting the edge joining the nodes n and $n + 1$ of the graph K_{n+1} , see Figure 11.

By the inductive hypothesis,

$$mgr(K_n) = (t - n + 1) mgr(K_{n-1}) (-p)^{n-1}$$

and the fact that K_n is the one point extension from K_{n-1} in the same way that H_{n+1} is the one point extension from K_n , see Figures (10) and (11), we have the following relation:

$$mgr(H_{n+1}) = (t - n + 1) mgr(K_n) (-p)^{n-1} \quad (19)$$

Then, by equations (18) and (19) we have:

$$\begin{aligned} K_{n+1} &= p ((-p)^{n-1} K_n + \dots) + (1 - p) ((t - n + 1) (-p)^{n-1} K_n + \dots) \\ &= (t - n) (-p)^n K_n + \dots \end{aligned}$$

where the dots denote terms whose degree is less than the edge number of K_{n+1} . We conclude that:

$$mgr(K_{n+1}) = (t - n) mgr(K_n) (-p)^n$$

which proves the claim. This recursive relation shows that $mgr(K_n)$ is non zero so it equals the highest degree term of the extended reliability polynomial of K_n :

$$K_n = \left(\prod_{\lambda=1}^{n-1} (t - \lambda) \right) (-p)^{(n-1)+(n-2)+\dots+1} + \dots$$

and this concludes the lemma. \square

Lemma 7.10. *Consider a partition \mathcal{A} with m classes: $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$. Then, the extended reliability polynomial of $K_n^{\mathcal{A}}$ is:*

$$R_t(K_n^{\mathcal{A}}) = \left(\prod_{\lambda=1}^{m-1} (t - \lambda) \right) (-p)^g + \dots$$

where the highest degree g of the expression equals edge number of $K_n^{\mathcal{A}}$ after taken out the irrelevant edges:

$$g = \sum_{i < j} |a_i| |a_j|$$

Proof: We will make the same abuse we did before identifying the extended reliability polynomial with its graph. The graph $K_n^{\mathcal{A}}$ has m nodes (these are the m classes of \mathcal{A}), $\binom{|a_i|}{2}$ irrelevant edges in each node i respectively and $|a_i||a_j|$ edges joining the nodes i and j .

Consider the graph $\bar{K}_n^{\mathcal{A}}$ resulting from deleting all the irrelevant edges of the graph $K_n^{\mathcal{A}}$. This way $K_n^{\mathcal{A}}$ and $\bar{K}_n^{\mathcal{A}}$ have the same extended reliability polynomial. By Lemma 7.8 we have the following relation between the graphs $\bar{K}_n^{\mathcal{A}}$ and K_m :

$$\begin{aligned} mgr(\bar{K}_n^{\mathcal{A}}) &= (-p)^{\sum_{i < j} (|a_i||a_j|-1)} mgr(K_m) \\ &= (-p)^{\sum_{i < j} (|a_i||a_j|-1)} \left(\prod_{\lambda=1}^{m-1} (t - \lambda) \right) (-p)^{\binom{m}{2}} \\ &= \left(\prod_{\lambda=1}^{m-1} (t - \lambda) \right) (-p)^{\sum_{i < j} |a_i||a_j|} \end{aligned}$$

and this concludes the proof. \square

Corollary 7.11. *The connectivity polynomial of the partition $\mathcal{A} \in Part_n$ is:*

$$\alpha_{\mathcal{A}}(t) = \prod_{\lambda=1}^{|\mathcal{A}|-1} (t - \lambda)$$

In particular, by the second item of Lemma 7.4 we have the following generator of Stirling numbers:

Corollary 7.12.

$$t^{n-1} = \sum_{m=1}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \prod_{\lambda=1}^{m-1} (t - \lambda)$$

Proof: By the second item of Lemma 7.4 and Corollary 7.11 we have the result. \square

This concludes the fourth step of the proof. By Proposition 7.6 and the above Corollary 7.11, we have finally proved the determinant formula 7.1. As an example, consider the $n = 4$ case. The partitions conjugation classes are:

$$\begin{aligned}
\mathcal{O}_1 &= \{ 1234 \} \\
\mathcal{O}_2 &= \{ 12 \overbrace{34}, 13 \overbrace{24}, 23 \overbrace{14}, 1 \overbrace{23} 4, \overbrace{13} 24, \overbrace{12} 34 \} \\
\mathcal{O}_3 &= \{ \overbrace{14} \overbrace{23}, \overbrace{13} \overbrace{24}, \overbrace{12} \overbrace{34} \} \\
\mathcal{O}_4 &= \{ 1 \overbrace{234}, 2 \overbrace{134}, 3 \overbrace{124}, \overbrace{123} 4 \} \\
\mathcal{O}_5 &= \{ \overbrace{1234} \}
\end{aligned}$$

Then we have that $|\mathcal{O}_i|$ equals 1, 6, 3, 4, 1 respectively. By the determinant formula 7.1 we conclude:

$$\begin{aligned}
\det(A(t)) &= [(t-3)(t-2)(t-1)] [(t-2)(t-1)]^6 [(t-1)]^3 [(t-1)]^4 \\
&= (t-3)(t-2)^7(t-1)^{14}
\end{aligned}$$

Appendix A. Spanning tree number

In this appendix we give an alternative spanning tree number factorization proof. The only non obvious argument that is worth to prove is Lemma Appendix A.3.

Consider a connected graph G such that there are connected subgraphs G_1 and G_2 only sharing vertices $\{1, 2, \dots, n\}$ with the property $G = G_1 \cup G_2$. In particular G_1 and G_2 have n distinguished vertices $\{1, 2, \dots, n\}$. We say that a subgraph $A \subset G$ is connected to the distinguished vertices if for every vertex $a \in V(A)$ there is a path in A connecting a with some distinguished vertex.

Lemma Appendix A.1. *\mathcal{T} is a spanning tree of G if and only if $\mathcal{T}_a = \mathcal{T} \cap G_a$ is a spanning forest of G_a connected to the distinguished vertices and $(\mathcal{P}(\mathcal{T}_1), \mathcal{P}(\mathcal{T}_2))$ is a connected and simply connected (CSC) pair.*

Definition Appendix A.1. Consider a partition $\mathcal{A} \in \text{Part}_n$. We define $\mathcal{F}(G, \mathcal{A})$ as the number of different spanning forest of G connected to the distinguished vertices with associated partition \mathcal{A} .

Denote by $\mathcal{S}(G)$ the spanning tree number of G . Then:

Corollary Appendix A.2.

$$\mathcal{S}(G) = \sum_{(\mathcal{A}, \mathcal{B}) \text{ is CSC}} \mathcal{F}(G_1, \mathcal{A}) \mathcal{F}(G_2, \mathcal{B})$$

Consider an ordering in $Part_n$ such that $i \leq j$ if \mathcal{A}_i is finer than \mathcal{A}_j . Define $\alpha_{ij} = \mathcal{F}(\hat{\mathcal{A}}_i, \mathcal{A}_j)$. Because $\mathcal{F}(\hat{\mathcal{A}}, \mathcal{B}) = 0$ if \mathcal{B} is not finer than \mathcal{A} and $\mathcal{F}(\hat{\mathcal{A}}, \mathcal{A}) = 1$ we have that

$$(\alpha_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{pmatrix}$$

In particular, (α_{ij}) is invertible and this result is independent of the chosen ordering in $Part_n$.

Lemma Appendix A.3.

$$\mathcal{S}(G^{\mathcal{B}}) = \sum_{(\mathcal{A}, \mathcal{B}) \text{ is CSC}} \mathcal{F}(G, \mathcal{A})$$

Proof: Applying simple factorization to every edge of an acyclic graph H is equivalent to consider a pair (H, ω) where $\omega \in \{0, 1\}^{E(H)}$ is a state such that $\omega(e) = 1$ means that the edge e is contracted and $\omega(e) = 0$ means that the edge e is deleted. We can identify every state ω with a spanning forest $F_\omega \subset H$ such that $V(F_\omega) = V(H)$ and $e \in E(F_\omega)$ if and only if $\omega(e) = 1$.

Consider the acyclic graph $\hat{\mathcal{A}}$. If the spanning forest F_ω is not connected to the distinguished vertices then $G * (\hat{\mathcal{A}}, \omega)$ is not connected and $\mathcal{S}(G * (\hat{\mathcal{A}}, \omega)) = 0$. On the contrary, if F_ω is connected to the distinguished vertices then there is a partition \mathcal{B} such that $G * (\hat{\mathcal{A}}, \omega) = G^{\mathcal{B}}$ and there are exactly $\alpha_{\mathcal{A}\mathcal{B}}$ of these states ω . We have proved that applying simple factorization to $\hat{\mathcal{A}}$ leads to the following result:

$$\mathcal{S}(G * \hat{\mathcal{A}}) = \sum_{\omega} \mathcal{S}(G * (\hat{\mathcal{A}}, \omega)) = \sum_{\mathcal{B}} \alpha_{\mathcal{A}\mathcal{B}} \mathcal{S}(G^{\mathcal{B}})$$

On the other hand, by corollary Appendix A.2 we have:

$$\mathcal{S}(G \cdot \hat{\mathcal{A}}) = \sum_{(\mathcal{C}, \mathcal{B}) \text{ is CSC}} \mathcal{F}(G, \mathcal{C}) \mathcal{F}(\hat{\mathcal{A}}, \mathcal{B}) = \sum_{(\mathcal{C}, \mathcal{B}) \text{ is CSC}} \alpha_{\mathcal{A}\mathcal{B}} \mathcal{F}(G, \mathcal{C})$$

Subtracting both identities we have:

$$\sum_{\mathcal{B}} \alpha_{\mathcal{AB}} \left(\mathcal{S}(G^{\mathcal{B}}) - \sum_{(\mathcal{C}, \mathcal{B}) \text{ is CSC}} \mathcal{F}(G, \mathcal{C}) \right) = 0$$

Because the matrix $(\alpha_{\mathcal{AB}})$ is invertible we have the result:

$$\mathcal{S}(G^{\mathcal{B}}) = \sum_{(\mathcal{C}, \mathcal{B}) \text{ is CSC}} \mathcal{F}(G, \mathcal{C})$$

□

Definition Appendix A.2. We define the simply connectivity matrix $A = (a_{\mathcal{AB}})$ such that $a_{\mathcal{AB}} = 1$ if $(\mathcal{A}, \mathcal{B})$ is a connected and simply connected pair and $a_{\mathcal{AB}} = 0$ if it is not.

Theorem Appendix A.4. *There is a matrix $B = (b_{\mathcal{AB}})$ such that*

$$\mathcal{S}(G) = \sum_{\mathcal{A}, \mathcal{B}} b_{\mathcal{AB}} \mathcal{S}(G_1^{\mathcal{A}}) \mathcal{S}(G_2^{\mathcal{B}})$$

if and only if $ABA = A$ where A is the simply connectivity matrix. In particular, if A is invertible then $B = A^{-1}$.

References

- [An] A. Andrzejak, *Splitting formulas for Tutte polynomials*, J. Combin. Theory Ser. B, 70 (1997), 346366.
- [Bi] N.L. Biggs, *Algebraic Graph Theory*, Cambridge, Cambridge University Press, 1993.
- [Bo] B. Bollobás, *Modern Graph Theory*, New York, Springer-Verlag, 1998.
- [Br] T. H. Brylawski, *A combinatorial model for series-parallel networks*, Transactions of the American Mathematical Society, vol. 154, 122, 1971.
- [BM] J.E. Bonin, A. de Mier, *Tutte polynomials of generalized parallel connections*, Adv. in Appl. Math., 32 (2004), 3143.

- [BMa] G.Birkhoff, S.MacLane, *A survey of modern algebra*, New York, The Mcmillan Company, 1965.
- [BO] T.H.Brylawski, J.G.Oxley, *The Tutte polynomial and its applications*, Matroid Applications (N. White ed.), Cambridge Univ. Press, Cambridge, 1992, pp.123-225.
- [BR] J.M.Burgos, F.Robledo *Factorization of network reliability with perfect nodes I: Introduction and statements*, Discrete Applied Mathematics, Vol.198, 2016
- [Bu] J.M.Burgos, *Factorization of network reliability with perfect nodes II: Connectivity matrix*, Discrete Applied Mathematics, Vol.198, 2016
- [Co] C.J.Colbourn, *The Combinatorics of Network Reliability*, New York, Oxford University Press, 1987.
- [FK] M.C.Fortuin, W.P.Kasteleyn, *On the random-cluster model: I. Introduction and relation to other models*, Physica (Elsevier) 57 (4), 536564, (1972)
- [Is] E.Ising, *Beitrag zur Theorie des Ferromagnetismus*, Z. Phys. 31, 253258, (1925).
- [Ja] F.Jaeger, D.Vertigan, D.J.A.Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc., 108 (1990), 3553.
- [Jo] V.F.R.Jones, *A polynomial invariant for knots via von Neumann algebra*, Bull. Amer. Math. Soc. (N.S.) 12, 103111, (1985).
- [Mo] F.Moskovitz and R.A.D.Center, *The analysis of redundancy networks*, Rome Air Development Center, Air Research and Development Center, United States Air Force, 1958.
- [Ne] S. Negami, *Polynomial invariants of graphs*, Trans. Amer. Math. Soc., 299 (1987), 601622.
- [On] L.Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition*, Physical Review, Series II 65 (34), 117149, (1944).

- [Po] R.B.Potts, *Some Generalized Order-Disorder Transformations* Mathematical Proceedings 48 (1), 106-109, (1952)
- [Ro] A.Rosenthal, *Computing the Reliability of Complex Networks*, Siam J.Applied Math., 32, No2, 384-393.
- [Sh] D.R.Shier, *Network reliability and algebraic structures*, New York, Oxford Clarendon Press, 1991.
- [Tr] L.Traldi, *On the colored Tutte polynomial of a graph of bounded treewidth*, Discrete Applied Mathematics 154 (2006), 1032-1036.
- [Tu] W.T.Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80-91.